



ON ROOT CLOSURE IN COMMUTATIVE RINGS

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الخلاصة :

ندرس في هذا البحث عدة شروط على الحلقة التبديلية R والتي لها علاقة بالجزور المغلقة . هذه الشروط تضمن شروط القسمة الداخلية على عناصر الحلقة R بدلاً من شروط الإغلاق على عناصر الحلقة الكسرية للحلقة R .

ABSTRACT

In this paper, we investigate several conditions on a commutative ring R which are related to root closure. These conditions involve internal divisibility conditions on elements of R rather than closedness conditions on elements of the total quotient ring of R .

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ON ROOT CLOSURE IN COMMUTATIVE RINGS

1. INTRODUCTION

Let R be a commutative ring with 1 and total quotient ring $T(R)$, and let $n \geq 1$ be an integer. Then R is n -root closed if whenever $x^n \in R$ for some $x \in T(R)$, then $x \in R$; and R is root closed if it is n -root closed for all $n \geq 1$. Also, R is $(2,3)$ -closed if whenever $x^2, x^3 \in R$ (equivalently, $x^n \in R$ for all sufficiently large integers $n \geq 1$) for some $x \in T(R)$, then $x \in R$. If R is n -root closed for some $n \geq 2$, then R is also $(2,3)$ -closed. However, even for integral domains, a $(2,3)$ -closed ring need not be n -root closed for any $n \geq 2$ [1, Example 2.3]. Following Swan [2], we say that R is *seminormal* if whenever $a^2 = b^3$ for $a, b \in R$, then $a = c^3$ and $b = c^2$ for some $c \in R$. Note that a seminormal ring is reduced. The importance of seminormality is that $\text{Pic}(R[X]) = \text{Pic}(R)$ if and only if $R/\text{Nil}(R)$ is seminormal [2, Theorem 1]. A seminormal ring is $(2,3)$ -closed, but not conversely (see Example 3.1). A reduced ring R with only a finite number of minimal prime ideals is $(2,3)$ -closed if and only if it is seminormal (cf. [3, Theorem 2]). In particular, an integral domain is seminormal if and only if it is $(2,3)$ -closed (cf. [4, Theorem 1.6]). For related results on seminormal rings and their relationship to $(2,3)$ -closedness, see [5, Section 3].

In [5], we investigated several internal divisibility conditions on a ring R related to seminormality. In this paper, we extend this study to ring-theoretic properties related to root closure. One of the central themes in this paper, and also in [5–9], is to generalize the study of certain “closedness” properties in integral domains to the context of commutative rings with zerodivisors and to replace these “closedness” conditions on elements of $T(R)$ by divisibility conditions on elements of R . Specifically, we replace $(a/b)^n \in R$ for $a \in R, b \in R - Z(R)$ by $b^n | a^n$ in R for $a, b \in R$, and we allow $b \in Z(R)$. Unlike the usual “closedness” properties, these new divisibility conditions need not hold for total quotient rings. Also, conditions which are equivalent for integral domains may be quite different for rings with zerodivisors. Many of these conditions force R to be reduced; so, as in [5], much of our emphasis is on subrings of direct products of integral domains.

In the second section, we relate additively regular rings to root closed, $(2,3)$ -closed, and integrally closed rings. The third section studies several conditions related to root closure. In the fourth section, we generalize and investigate the concepts of strongly radical ideal and rooty ring from the integral domain case, as in [10–12], to rings with zerodivisors. The final section gives another generalization of root closed rings, strongly radical ideals, and rooty rings by replacing $(a/b)^n \in R$ for $a \in R, b \in R - Z(R)$ by $b^n | a^n$ in R for $a \in R, b \in R - \text{Nil}(R)$.

Throughout, all rings will be commutative with $1 \neq 0$. When we say that a ring A is a subring of a ring B , we mean that A and B have the same identity element. For a ring R , we denote its set of zerodivisors by $Z(R)$, its ideal of nilpotent elements by $\text{Nil}(R)$, its group of units by $U(R)$, its total quotient ring by $T(R)$, and its integral closure (in $T(R)$) by R' . As usual, $x \in R$ is called a regular element if x is not a zerodivisor of R . Also, X and Y will always denote indeterminates. Any undefined terminology or notation is standard, as in [13–15]. See [16] for a recent survey article on root closure in commutative rings.

2. ADDITIVELY REGULAR RINGS AND ROOT CLOSURE

A ring R is called *additively regular* if for each $x \in T(R)$, there is a $y \in R$ such that $x + y$ is a regular element of $T(R)$. If $Z(R)$ is a finite union of prime ideals, then R is additively regular [14, Theorem 7.2]. In particular, Noetherian rings and reduced rings with only a finite number of minimal prime ideals are additively regular. We start with the following result.

Lemma 2.1. *The following statements are equivalent for a ring R .*

- (1) R is additively regular.
- (2) For each $a \in R, b \in R - Z(R)$, and $n \geq 1$, there is a $y \in R$ such that $a + yb^n$ is a regular element of R .

Proof.

(1) \Rightarrow (2): Let $a \in R, b \in R - Z(R)$, and $n \geq 1$. Since $a/b^n \in T(R)$ and R is additively regular, $a/b^n + y$ is a regular element of $T(R)$ for some $y \in R$. Hence $a + yb^n = (a/b^n + y)b^n$ is a regular element of R .

(2) \Rightarrow (1): Let $x = a/b \in T(R)$ for some $a \in R$ and $b \in R - Z(R)$. Then $a + yb$ is a regular element of R for some $y \in R$. Thus $x + y = a/b + y = (a + yb)/b$ is a regular element of $T(R)$. \square

Our next three results show that if R is additively regular (for example, Noetherian), then R is root (resp., n -root, (2,3)-, integrally) closed if and only if R is root (resp., n -root, (2,3)-, integrally) closed for the group of units of $T(R)$. The (2,3)-closed case has been studied in [5, Section 3]; in particular, the (2,3)-closed case of Proposition 2.5 follows from [5, Theorem 3.3].

Proposition 2.2. *The following statements are equivalent for an additively regular ring R .*

- (1) R is root closed.
- (2) If $x^n \in R$ for some $x \in U(T(R))$ and $n \geq 1$, then $x \in R$.

Proof.

(1) \Rightarrow (2) is clear.

(2) \Rightarrow (1): Let $x = a/b \in T(R)$ for some $a \in R$ and $b \in R - Z(R)$. Suppose that $x^n = a^n/b^n \in R$ for some $n \geq 1$. Since R is additively regular, $a + yb^n$ is a regular element of R for some $y \in R$ by Lemma 2.1. Hence $c = (a + yb^n)/b \in U(T(R))$. Since $a^n/b^n \in R$, it is easy to verify that $c^n \in R$. Since $c \in U(T(R))$ and $c^n \in R$, by hypothesis $c = (a + yb^n)/b = a/b + yb^{n-1} \in R$. Thus $x = a/b \in R$. \square

Proposition 2.3. *The following statements are equivalent for an additively regular ring R .*

- (1) R is (2,3)-closed.
- (2) If $x^2, x^3 \in R$ for some $x \in U(T(R))$, then $x \in R$.

Proof.

(1) \Rightarrow (2) is clear.

(2) \Rightarrow (1): Let $x = a/b \in T(R)$ for some $a \in R$ and $b \in R - Z(R)$. Suppose that $x^2, x^3 \in R$. Since R is additively regular, $a + yb^2$ is a regular element of R for some $y \in R$ by Lemma 2.1. Hence $c = (a + yb^2)/b \in U(T(R))$. Since $a^2/b^2, a^3/b^3 \in R$, it is easy to verify that $c^2, c^3 \in R$. Since $c \in U(T(R))$ and $c^2, c^3 \in R$, by hypothesis $c = (a + yb^2)/b = a/b + yb \in R$. Thus $x = a/b \in R$. \square

Proposition 2.4. *The following statements are equivalent for an additively regular ring R .*

- (1) R is integrally closed.
- (2) If $x \in R'$ for some $x \in U(T(R))$, then $x \in R$.

Proof.

(1) \Rightarrow (2) is clear.

(2) \Rightarrow (1): Let $x \in R'$. Since R is additively regular, $x + y$ is a regular element of $T(R)$ for some $y \in R$. Thus $x + y \in U(T(R))$, and $x + y \in R'$ since $x \in R'$ and $y \in R \subset R'$. Hence $x + y \in R$ by hypothesis, and thus $x \in R$. \square

Let $D = \prod_{i \in I} D_i$, where each D_i is an integral domain, and let R be a subring of D . As in [5, Section 2], we say that $y = (y_i) \in R$ is an *extension* of (or *extends*) $x = (x_i) \in R$, denoted by yEx , if $y_i = x_i$ whenever $x_i \neq 0$. Recall that a commutative ring R is a subring of a direct product of integral domains if and only if it is reduced,

and that R is a subring of a direct product of finitely many integral domains if and only if it is reduced with only a finite number of minimal prime ideals (this observation gives another proof of Corollary 2.6).

Proposition 2.5. *Let R be a subring of a direct product of integral domains. Suppose that each nonzero element of R can be extended to a regular element of R . Then R is additively regular, and therefore R is root (resp., n -root, (2,3)-, integrally) closed if and only if R is root (resp., n -root, (2,3)-, integrally) closed for the group of units of $T(R)$.*

Proof. Let $x \in T(R)$. We may assume that $x \neq 0$. Then $x = a/b$ for some $0 \neq a = (a_i) \in R$ and $b = (b_i) \in R - Z(R)$. By hypothesis, there is a regular element $y = (y_i) \in R$ such that yEa . We show that $a + b(y - a)$ is a regular element of R . If not, then $c(a + b(y - a)) = 0$ for some $0 \neq c = (c_i) \in R$. Thus $c_i = 0$ whenever $a_i \neq 0$, and hence $ca = 0$. Thus $cb(y - a) = 0$, and hence $c(y - a) = 0$ since b is regular. Thus $cy = c((y - a) + a) = c(y - a) + ca = 0$, a contradiction since y is regular. Hence $a + b(y - a)$ is a regular element of R , and thus $a/b + (y - a) = (a + b(y - a))/b$ is a regular element of $T(R)$. Hence R is additively regular. The "therefore" statement follows from Propositions 2.2, 2.3, and 2.4. \square

In particular, our next corollary applies if R is a reduced ring with only a finite number of minimal prime ideals (for example, if R is a reduced Noetherian ring). The (2,3)-closed case is also in [5, Corollary 3.2].

Corollary 2.6. *Let R be a subring of a direct product of finitely many integral domains. Then R is additively regular, and therefore R is root (resp., n -root, (2,3)-, integrally) closed if and only if R is root (resp., n -root, (2,3)-, integrally) closed for the group of units of $T(R)$.*

Proof. Since each nonzero element of R can be extended to a regular element of R when R is a subring of a direct product of finitely many integral domains by [5, Lemma 2.5], we are done by Proposition 2.5. \square

Question 2.7. Do Propositions 2.2, 2.3, and 2.4 hold without the additively regular hypothesis?

3. REDUCED RINGS AND ROOT CLOSURE

In this section, we consider several conditions on a commutative ring R which are related to root closure. Let $n \geq 1$ be an integer. We call R *strongly n -root closed* if whenever $b^n|a^n$ for some $a, b \in R$, then $b|a$; and we say that R is *strongly root closed* if it is strongly n -root closed for all $n \geq 1$. We call R *strongly (2,3)-closed* if whenever $b^2|a^2$ and $b^3|a^3$ (equivalently, $b^n|a^n$ for all sufficiently large integers $n \geq 1$) for some $a, b \in R$, then $b|a$. In each case, we have just replaced $(a/b)^n \in R$ for $a \in R, b \in R - Z(R)$ by $b^n|a^n$ in R , and we have allowed $b \in Z(R)$. These definitions are equivalent to the earlier definitions when R is an integral domain.

It is easy to show that a direct product $R = \prod_{i \in I} R_i$ of rings is strongly root (resp., n -root, (2,3)-) closed if and only if each R_i is strongly root (resp., n -root, (2,3)-) closed. Also, if R is strongly root (resp., n -root, (2,3)-) closed, then R_S is also strongly root (resp., n -root, (2,3)-) closed for any multiplicatively closed set $S \subset R$.

Clearly a strongly root (resp., n -root, (2,3)-) closed ring is root (resp., n -root, (2,3)-) closed, and a strongly n -root closed ring (for some $n \geq 2$) is also strongly (2,3)-closed (however, the converse fails even for integral domains). Observe that if a ring R is strongly (2,3)-closed (or strongly n -root closed for some $n \geq 2$), then R must be reduced. However, a (2,3)-closed ring need not be reduced. The following two examples show that even for reduced rings, a root (resp., (2,3)-) closed ring need not be strongly root (resp., (2,3)-) closed. The first example is a subring of a direct product of infinitely many integral domains (cf. [5, Example 3.5]). The second, more complicated example, is a subring of a direct product of finitely many integral domains.

Example 3.1. Let $n > 1$. Then there is a reduced, non-seminormal ring R with infinitely many minimal prime ideals such that R is root closed (and hence n -root closed and (2,3)-closed) and $b^m|a^m$ for some $a, b \in Z(R)$ and all $m \geq n$, but $b \nmid a$ in R (i.e., R is neither strongly n -root closed nor strongly (2,3)-closed).

Proof. Let K be a field and $n > 1$. Let $R = K(1, 1, 1, \dots) + \bigoplus_{i \geq 1} X^n K[X] \subset \prod_{i \geq 1} K[X]$. It is easy to verify that R is not seminormal and that $T(R) = R$, and hence R is root closed. Let $a = (X^{n+1}, 0, 0, \dots)$, $b = (X^n, 0, 0, \dots) \in R$. Then $b^m | a^m$ for all $m \geq n$, but $b \nmid a$ since $(X, f_2, f_3, \dots) \notin R$ for any $\{f_i\}_{i \geq 2} \subset K[X]$. We next show that R has infinitely many minimal prime ideals. For each $j \geq 1$, let $P_j = \{y \in \bigoplus_{i \geq 1} X^n K[X] \mid \text{the } j\text{-th-coordinate of } y \text{ is zero}\}$. It is easy to verify that each P_j is a prime ideal of R . Also, each P_j is a minimal prime ideal of R by [14, Corollary 2.2]. Hence $\{P_i\}_{i \geq 1}$ is an infinite set of minimal prime ideals of R . \square

Example 3.2. Let $n > 1$. Then there is a reduced seminormal ring R with finitely many minimal prime ideals such that R is root closed (and hence n -root closed and (2,3)-closed) and $b^n | a^n$ for some $a, b \in Z(R)$, but $b \nmid a$ in R (i.e., R is not strongly n -root closed).

Proof. Let $n > 1$ and let $D = \mathbb{Z}_2[X, Y] \oplus \mathbb{Z}_2[X, Y]$. Let R be the subring of D generated by $\{(1, 1), (X^n, Y), (X^{n-1}Y, 0), (X^{n-2}Y, 0), \dots, (XY, 0), (Y, 0), (0, X)\}$. First, we make the following remarks:

- (1) $(Y^m, 0) \in R$ for each $m \geq 1$.
- (2) $(0, X^m) \in R$ for each $m \geq 1$.
- (3) $(0, X^m Y^s) \in R$ for each $m \geq 1$ and $s \geq 1$.
- (4) $(X^m Y^s, 0) \in R$ for each $m \geq 1$ and $s \geq 1$. To see this: Let $m = kn + r$ with $k \geq 0$ and $1 \leq r \leq n - 1$. Then $(X^m Y^s, 0) = (X^n, Y)^k (X^r Y, 0) (Y, 0)^{s-1} \in R$.
- (5) $(X^m, 0) \notin R$ for each $m \geq 1$.
- (6) $(0, Y^m) \notin R$ for each $m \geq 1$.
- (7) If $(h, f) \in R$ for some $h, f \in \mathbb{Z}_2[X, Y]$, then $(X^n)^m$ is a term of h for some $m \geq 1$ if and only if Y^m is a term of f .

We first show that R is not strongly n -root closed (and hence R is not strongly root closed). Let $a = (XY, 0)$, $b = (Y, 0) \in R$. Then $b^n | a^n$, but $b \nmid a$ since $(X, d) \notin R$ for any $d \in \mathbb{Z}_2[X, Y]$. We next show that R is root closed (and hence R is n -root closed, (2,3)-closed, and seminormal [5, Corollary 3.2]). Since R is additively regular by Corollary 2.6, we need only show that if $a, b \in R - Z(R)$ and $b^m | a^m$ for some $m > 1$, then $b | a$ in R . Let $a = (h_1, f_1)$, $b = (h_2, f_2) \in R - Z(R)$ such that $(h_1, f_1)^m = (h_2, f_2)^m (h_3, f_3)$ for some $(h_3, f_3) \in R$ and $m > 1$. Observe that h_1, f_1, h_2, f_2, h_3 , and f_3 are nonzero elements of $\mathbb{Z}_2[X, Y]$. Hence $h_3 = c_1 + c_2 + c$ for some $c_1 \in \mathbb{Z}_2[Y, XY, X^2Y, X^3Y, \dots]$, $c_2 \in \mathbb{Z}_2[X^n]$, and $c \in \mathbb{Z}_2$ with $c_1(0, 0) = c_2(0) = 0$. Also, $f_3 = d_1 + d_2 + c$ for some $d_1 \in \mathbb{Z}_2[X, YX, Y^2X, Y^3X, \dots]$ and $d_2 \in \mathbb{Z}_2[Y]$ with $d_1(0, 0) = d_2(0) = 0$. Let $\varphi: \mathbb{Z}_2[X] \rightarrow \mathbb{Z}_2[Y^{1/n}]$ be the ring isomorphism determined by $\varphi(1) = 1$ and $\varphi(X) = Y^{1/n}$. Observe that $\varphi(c_2) = d_2$, by remark (7). Also, observe that if $\varphi(f) \in \mathbb{Z}_2[Y]$, then $f \in \mathbb{Z}_2[X^n]$, by definition of φ . Since D is integrally closed, $(h_1, f_1) = (h_2, f_2)(h_4, f_4)$ for some $(h_4, f_4) \in D$. Hence $h_4 = e_1 + e_2 + c$ for some $e_1 \in \mathbb{Z}_2[Y, XY, X^2Y, X^3Y, \dots]$ and $e_2 \in \mathbb{Z}_2[X]$ with $e_1(0, 0) = e_2(0) = 0$. Also, $f_4 = w_1 + w_2 + c$ for some $w_1 \in \mathbb{Z}_2[X, YX, Y^2X, Y^3X, \dots]$ and $w_2 \in \mathbb{Z}_2[Y]$ with $w_1(0, 0) = w_2(0) = 0$. We need to show that $(h_4, f_4) \in R$. Since $(e_1, 0) \in R$, by remarks (1) and (4), and $(0, w_1) \in R$, by remarks (2) and (3), we have $(e_1, w_1) \in R$. Thus we need only show that $(e_2 + c, w_2 + c) \in R$. Since $h_4^m = h_3$ and $e_1 \in \mathbb{Z}_2[Y, XY, X^2Y, \dots]$, $e_1(0, 0) = 0$, $e_2 \in \mathbb{Z}_2[X]$, and $c_2 \in \mathbb{Z}_2[X^n]$, we have $(e_2 + c)^m = c_2 + c$. Also, since $f_4^m = f_3$ and $w_1 \in \mathbb{Z}_2[X, YX, Y^2X, \dots]$, $w_1(0, 0) = 0$, $w_2 \in \mathbb{Z}_2[Y]$, and $d_2 \in \mathbb{Z}_2[Y]$, we have $(w_2 + c)^m = d_2 + c$. Since $(\varphi(e_2 + c))^m = \varphi((e_2 + c)^m) = \varphi(c_2 + c) = d_2 + c = (w_2 + c)^m$ and $\mathbb{Z}_2[Y^{1/n}]$ is a UFD with $U(\mathbb{Z}_2[Y^{1/n}]) = \{1\}$, we have $\varphi(e_2 + c) = w_2 + c \in \mathbb{Z}_2[Y]$. Hence $e_2 + c \in \mathbb{Z}_2[X^n]$, and thus $(e_2 + c, w_2 + c) \in R$. \square

In view of Examples 3.1 and 3.2, we have the following special cases where a root closed ring is strongly root closed.

Proposition 3.3. *Let R be a subring of a direct product of integral domains such that each nonzero element of R can be extended to a regular element of R . Furthermore, suppose that if $x \in U(T(R))$, then $x^m \in R$ for some $m \geq 1$. Then R is strongly root closed if and only if R is root closed (if and only if $U(T(R)) = U(R)$, i.e., $T(R) = R$).*

Proof. We need only show that R is strongly root closed if R is root closed. Suppose that R is root closed and $b^n|a^n$ for some nonzero $a, b \in Z(R)$ and $n \geq 1$. Then $a^n = b^n k$ for some $k \in R$. Let d and z be regular elements of R such that dEb and zEa . Hence $a = b((d^{n-1}/z^{n-1})k)$. Since $d/z \in U(T(R))$, $(d/z)^m \in R$ for some $m \geq 1$, and R is root closed, we have $d/z \in R$. Hence $b|a$ in R . The equivalence of the statement in parentheses follows from Proposition 2.2. \square

Corollary 3.4. *Let R be a subring of a direct product of finitely many integral domains. Suppose that if $x \in U(T(R))$, then $x^m \in R$ for some $m \geq 1$. Then R is strongly root closed if and only if R is root closed (if and only if $U(T(R)) = U(R)$, i.e., $T(R) = R$).*

Proof. By [5, Lemma 2.5], every nonzero element of R can be extended to a regular element of R . Hence the proof is completed by Proposition 3.3. \square

The following results are immediate consequences of Proposition 3.3 or Corollary 3.4; so we state them without proof. An important special case of Corollary 3.6 is when R is reduced with only a finite number of minimal prime ideals (for example, when R is reduced and Noetherian) and $U(T(R))$ is a torsion group. Corollary 3.7 also follows from our earlier observation that a direct product of strongly root closed rings is strongly root closed (cf. Corollary 4.7).

Corollary 3.5. *Let R be a subring of a direct product of integral domains such that each nonzero element of R can be extended to a regular element of R (for example, if R is a direct product of finitely many integral domains). If $T(R) = R$, then R is strongly root closed.*

Corollary 3.6. *Let R be a subring of a direct product of integral domains such that each nonzero element of R can be extended to a regular element of R (for example, if R is a direct product of finitely many integral domains). If $U(T(R))$ is a torsion group, then R is strongly root closed if and only if R is root closed (if and only if $T(R) = R$).*

Corollary 3.7. *Let R be a subring of a direct product of finitely many finite fields (i.e., R is a reduced finite ring). Then R is strongly root closed.*

Let R be a subring of a direct product $D = \prod_{i \in I} D_i$ of integral domains D_i , and let $x = (x_i), y = (y_i) \in R$. Recall from [5, Section 2] that x and y are of the same type, denoted by $x \sim y$, if $x_i = 0 \Leftrightarrow y_i = 0$ for each $i \in I$. Our next result is the "root closed" analog of [5, Theorem 2.2].

Proposition 3.8. *Let R be a subring of a direct product of integral domains such that each nonzero element of R can be extended to a regular element of R (for example, if R is a subring of a direct product of finitely many integral domains). Suppose that if $b^n|a^n$ for some $a, b \in Z(R)$ and $n > 1$, then $a^n = b^n k$ for some $k \in R$ such that $k \sim a$. Then R is strongly root (resp., (2,3)-) closed if and only if R is root (resp., (2,3)-) closed.*

Proof. We need only show that R is strongly root closed if R is root closed. Suppose that $b^n|a^n$ for some nonzero $a, b \in Z(R)$ and $n > 1$. Then $a^n = b^n k$ for some $k \in R$ such that $k \sim a$. Let d be a regular element of R such that dEb . Then $a^n = d^n k$ since dEb and $k \sim a$. Hence $a = dc$ for some $c \in R$ since R is root closed. Thus $a = bc$, so $b|a$. The proof for the (2,3)-closed case is similar, and hence will be omitted. \square

We end this section with the following remark which shows that the hypothesis that each nonzero element of R can be extended to a regular element of R is independent of the equivalence of the root closed and strongly root closed hypotheses.

Remark 3.9.

- (a) Let R be as in Example 3.1 with $n = 2$. Then $T(R) = R$, and thus R is root closed, but R is neither strongly root closed nor seminormal, and some nonzero element of R cannot be extended to a regular element of R .

- (b) Let K be a field and $R = K(1, 1, 1, \dots) + \bigoplus_{i \geq 1} XK[X] \subset \prod_{i \geq 1} K[X]$. Then $T(R) = R$, and R is strongly root closed (and hence strongly (2,3)-closed), but some nonzero element of R cannot be extended to a regular element of R .
- (c) Let R be as in Example 3.2. Then each element of R can be extended to a regular element of R and R is root closed, but R is not strongly root closed.
- (d) It is well known that a (n -, for $n \geq 2$) root closed integral domain is seminormal. More generally, if a ring R is strongly root closed (and hence root closed) and R has finitely many minimal prime ideals, then R is seminormal by [5, Corollary 3.2]. However, let R (with $K = \mathbb{Z}_2$) be as in [5, Example 2.7(b)] (see Example 4.1). Then R is not seminormal, but it is easy to verify that R is strongly root closed (and hence also strongly (2,3)-closed).

4. STRONGLY ROOTY RINGS AND STRONGLY ROOT CLOSED RINGS

In this section, we generalize the “closedness” concepts of strongly radical ideal and rooty ring for integral domains, as in [10–12], to internal divisibility concepts in rings with zerodivisors.

Let I be an ideal of an integral domain R with quotient field K . In [10], I was called *strongly radical* if whenever $x^n \in I$ for some $x \in K$ and $n \geq 1$, then $x \in I$. In [11], I was called *boldly radical* if whenever $x^n \in I$ for some $x \in K$ and all sufficiently large integers $n \geq 1$ (equivalently, $x^2, x^3 \in I$), then $x \in I$. Also, R was called *rooty* if each radical ideal of R is strongly radical. A root closed integral domain is certainly also rooty, but not conversely (see the paragraph preceding Proposition 4.3). Moreover, R is (2,3)-closed (resp., rooty) if and only if each (maximal) prime ideal of R is boldly (resp., strongly) radical [11, Theorem 1.3 and Remarks 1.2.2 and 1.2.3]. These concepts were defined for arbitrary rings in [11], but most of the results were stated for integral domains. For other characterizations of strongly radical and boldly radical ideals, see [12, Theorems 1.1 and 1.2].

In this paper, we call an ideal I of a ring R *strongly radical* if whenever $a^n \in b^n I$ for some $a, b \in R$ and $n \geq 1$, then $a \in bI$. If every radical ideal of R is strongly radical, then R is called a *strongly rooty ring*. We call I *boldly radical* if whenever $a^n \in b^n I$ for all sufficiently large integers $n \geq 1$ (equivalently, $a^2 \in b^2 I, a^3 \in b^3 I$) for some $a, b \in R$, then $a \in bI$. These “new” definitions are equivalent to our earlier definitions in the integral domain case. Clearly, strongly radical and boldly radical ideals are radical ideals, and a strongly radical ideal is boldly radical, but a boldly radical ideal need not be strongly radical.

Observe that an integral domain R is a strongly rooty ring if and only if R is a rooty ring, and that if a ring R admits a strongly (or boldly) radical ideal, then R must be reduced. We have already observed that a root closed ring is a rooty ring. The following is an example of a reduced ring R which is a strongly root closed ring (and hence R is root closed, and also a rooty ring), but R is not a strongly rooty ring.

Example 4.1. Let $R = \mathbb{Z}_2(1, 1, 1, \dots) + \bigoplus_{n \geq 1} X^2 \mathbb{Z}_2[X] + \{\alpha_1 y_1 + \dots + \alpha_n y_n \mid \alpha_i \in \mathbb{Z}_2 \text{ and } n \geq 1\} \subset \prod_{n \geq 1} \mathbb{Z}_2[X] = T$, where $y_n = X(e_n + e_{n+1})$ and $e_m \in T$ has a 1 in the m th slot and 0 elsewhere (this is [5, Example 2.7(b)] with $K = \mathbb{Z}_2$). Then R is not seminormal, $T(R) = R$ (and hence R is root closed and rooty), and it is easy to verify that R is a strongly root closed ring. Let $c = (X^2, 0, 0, \dots) \in R$. Then $I = cR$ is a radical ideal of R . Now, let $b = c$ and $a = (X^3, 0, 0, \dots) \in R$. Then $a^2 \in b^2 I, a^3 \in b^3 I$, but $a \notin bI$ since $(X, f_2, f_3, \dots) \notin I$ for any $\{f_n\}_{n \geq 2} \subset \mathbb{Z}_2[X]$. Hence I is not boldly radical, and thus not strongly radical. Hence R is not a strongly rooty ring.

In the following result, we show that a strongly rooty ring is seminormal. Example 4.1 above shows that a strongly root closed (and hence strongly (2,3)-closed) ring need not be seminormal. Although a seminormal ring is (2,3)-closed, we do not know if a seminormal ring is necessarily strongly (2,3)-closed.

Proposition 4.2. *Let R be a ring such that every radical ideal of R is boldly radical. Then R is seminormal (and hence (2,3)-closed). In particular, a strongly rooty ring is seminormal.*

Proof. Since R is reduced, we may view R as a subring of a direct product of integral domains. Suppose that $a^2 = b^3$ for some $a, b \in R$. Hence $a \sim b$ and $a^2 \in b^2 \text{Rad}(bR), a^3 \in b^3 \text{Rad}(bR)$. Thus $a = bk$ for some $k \in \text{Rad}(bR)$. Since $a \sim b, k \in \text{Rad}(bR)$, and $a = bk$, we have $k \sim a$. Thus R is seminormal by [5, Theorem 2.2]. \square

By [11, Example 2.1], a (2,3)-closed integral domain need not be a (strongly) rooty ring. Also, by [11, Example 1.6], a quasilocal integral domain which is a (strongly) rooty ring need not be a (strongly) root closed ring. (A simpler such example, namely $R = \mathbb{R} + X\mathbb{C}[X]$, is given in [12, Example 2.13].) See [12] for other results on quasilocal rooty domains. We ask the reader to compare the following result with [11, Proposition 1.7] and [12, Theorem 2.1].

Proposition 4.3. *Let R be a strongly rooty ring. If R is not quasilocal, then R is a strongly root closed ring (and hence R is root closed).*

Proof. Let M and N be distinct maximal ideals of R . Let $a \in M, b \in N$ such that $a + b = 1$. Suppose that $c^n = d^n k$ for some $c, d, k \in R$, and $n \geq 1$. Hence $(ca)^n = d^n ka^n$ and $(cb)^n = d^n kb^n$. Since M and N are each strongly radical and $a \in M, b \in N$, we have $ca = df$ and $cb = dh$ for some $f \in M, h \in N$. Thus $c = c(a + b) = ca + cb = df + dh = d(f + h)$. Hence R is strongly root closed. \square

If I is a strongly (resp., boldly) radical ideal of R in the sense of [10–12], and J is a radical ideal of R such that $J \subset I$, then J is strongly (resp., boldly) radical in the sense of [10–12] (see [11, Remarks 1.2.3 and 1.2.2]). The following is an example of a ring R that admits a strongly (resp., boldly) radical ideal I such that J is a radical ideal of R and $J \subset I$, but J is not strongly (resp., boldly) radical.

Example 4.4. Let R be as in Example 4.1. Then R is a quasilocal ring with maximal ideal $Z(R)$. It is easy to check that $I = Z(R)$ is a strongly (resp., boldly) radical ideal of R , but $J = (X^2, 0, 0, \dots)R$ is a radical ideal of R which is not strongly (resp., boldly) radical by Example 4.1

If R is an integral domain with quotient field K , then an ideal I of R is a strongly radical ideal of R if and only if I is a radical ideal of the overring $I : I = \{x \in K | xI \subset I\}$ and $I : I$ is root closed (see [10, Proposition 1.4], [12, Theorem 1.1]). The following is an example of a ring R that admits a radical ideal I such that I is a radical ideal of $I : I = \{x \in T(R) | xI \subset I\}$ and $I : I$ is a strongly root closed ring (and hence root closed), but I is not a strongly radical ideal of R .

Example 4.5. Let R be as in Example 4.1 and $I = (X^2, 0, 0, \dots)R$. Then I is a radical ideal of R and since $I : I = R, I$ is a radical ideal of $I : I$. Also, $I : I = R$ is a strongly root closed ring (and hence is root closed), but I is not a strongly radical ideal of R by Example 4.1.

We end this section by showing that the concepts introduced in this and the previous section are all equivalent for reduced zero-dimensional commutative rings. Recall that a ring R is *von Neumann regular* if for each $a \in R$, there is a $b \in R$ with $aba = a$; equivalently, if and only if R is reduced and zero-dimensional [14, Remark, page 5]. It is well known that a von Neumann regular ring R is root closed (since $T(R) = R$) and seminormal. Example 4.1 shows that our next result does not extend to rings with $\dim(R) \geq 1$. Also, conditions (2)–(6) of Proposition 4.6 are not equivalent for one-dimensional integral domains (see [16, Examples 7.1]).

Proposition 4.6. *The following statements are equivalent for a zero-dimensional ring R .*

- (1) R is reduced (i.e., von Neumann regular).
- (2) R is seminormal.
- (3) R is strongly rooty.
- (4) R is strongly root closed.
- (5) R is strongly n -root closed for some integer $n \geq 2$.
- (6) R is strongly (2,3)-closed.

Proof. We have already observed that any of conditions (2)–(6) implies that R is reduced, that a von Neumann regular ring is seminormal, and that (4) \Rightarrow (5) \Rightarrow (6). Note that (3) \Rightarrow (4) by Proposition 4.3, since a reduced zero-dimensional quasilocal ring is a field; thus we need only show that (1) \Rightarrow (3).

(1) \Rightarrow (3): Let I be a radical ideal of a von Neumann regular ring R , and suppose that $a^n \in b^n I$ for some $a, b \in R$ and $n \geq 1$. By [14, Corollary 3.3], $a = eu$ and $b = fv$ for some idempotents $e, f \in R$ and $u, v \in u(R)$. Thus $eu^n \in fv^n I = fI = bI$, and hence $a = eu \in bI$. Thus I is a strongly radical ideal, and hence R is strongly rooty. \square

Corollary 4.7. *A finite ring R is seminormal (resp., strongly rooty, strongly root closed, strongly n -root closed for some $n \geq 2$, strongly (2,3)-closed) if and only if it is reduced (i.e., a direct product of finitely many finite fields).*

Corollary 4.8. *Let $n \geq 2$. Then \mathbb{Z}_n is seminormal (resp., strongly rooty, strongly root closed, strongly d -root closed for some $d \geq 2$, strongly (2,3)-closed) if and only if n is square-free.*

5. ALMOST STRONGLY ROOTY RINGS

In this section, we give another generalization of root closure for rings with zerodivisors. In these definitions, we replace $b^n|a^n$ for $a, b \in R$ by $b^n|a^n$ for $a \in R, b \in R - Nil(R)$.

Let $n \geq 1$ be an integer. A ring R is said to be *almost strongly n -root closed* if whenever $b^n|a^n$ for some $a \in R, b \in R - Nil(R)$, then $b|a$; and R is *almost strongly root closed* if it is almost strongly n -root closed for all $n \geq 1$. We define R to be *almost strongly (2,3)-closed* if whenever $b^2|a^2$ and $b^3|a^3$ (equivalently, $b^n|a^n$ for all sufficiently large integers $n \geq 1$) for some $a \in R, b \in R - Nil(R)$, then $b|a$. An ideal I of R is called *almost strongly radical* if whenever $a^n \in b^n I$ for some $a \in R, b \in R - Nil(R)$, and $n \geq 1$, then $a \in bI$; and I is called *almost boldly radical* if whenever $a^n \in b^n I$ for all sufficiently large integers $n \geq 1$ (equivalently, $a^2 \in b^2 I$ and $a^3 \in b^3 I$) for some $a \in R, b \in R - Nil(R)$, then $a \in bI$. If every radical ideal of R is almost strongly radical, then R is called an *almost strongly rooty ring*. Clearly, almost strongly and almost boldly radical ideals are radical ideals, and an almost strongly radical ideal is almost boldly radical. Also, recall that an ideal I of R is called *divided* if I is comparable, under inclusion, to every principal ideal of R .

Clearly, an almost strongly root closed (resp., almost strongly n -root closed, almost strongly (2,3)-closed, almost strongly rooty) ring is a root closed (resp., n -root closed, (2,3)-closed, rooty) ring.

A strongly root closed (resp., strongly n -root closed, strongly (2,3)-closed, strongly rooty) ring is an almost strongly root closed (resp., almost strongly n -root closed, almost strongly (2,3)-closed, almost strongly rooty) ring, and these concepts agree for reduced rings. However, an almost strongly root closed (resp., almost strongly n -root closed, almost strongly (2,3)-closed, almost strongly rooty) ring need not be a strongly root closed (resp., strongly n -root closed, strongly (2,3)-closed, strongly rooty) ring (see Remark 5.17(a)).

A direct product of strongly (n -)root (resp., strongly (2,3)-) closed rings is again strongly (n -)root (resp., strongly (2,3)-) closed; however, this need not hold for almost strongly (n -)root (resp., almost strongly (2,3)-) closed rings (see Corollary 5.16).

If R is an integral domain, then $Nil(R) = \{0\}$ is a divided ideal. Thus, to generalize concepts from integral domains to rings with zerodivisors, it is natural to assume that $Nil(R)$ is a divided ideal and replace $b \neq 0$ by $b \notin Nil(R)$. We will see that the divisibility conditions we have defined in this section all force $Nil(R)$ to be divided. An important consequence of our next result is that if $Nil(R)$ is nonzero and divided, then it is the unique minimal prime ideal of R .

Proposition 5.1. *A nonzero divided radical ideal I of a ring R is a prime ideal of R . In particular, if $Nil(R)$ is a nonzero divided ideal of R , then $Nil(R)$ is a prime ideal of R .*

Proof. Suppose that I is not prime. Then there are $x, y \in R - I$ with $xy \in I$. Thus $x^2 \notin I$, and hence $I \subset x^2R$, since I is a divided radical ideal. Thus $xy = x^2d$ for some $d \in R$, and hence $y - xd \notin I$, since $xd \in I$ and $y \notin I$. Thus $I \subset (y - xd)R$, since I is divided, and hence $xI \subset x(y - xd)R = \{0\}$. Let $0 \neq i \in I \subset x^2R$. Then $i = x^2r$ for some $r \in R$, and $xr \in I$. Thus $i = x(xr) = 0$, a contradiction. Hence I must be a prime ideal of R . \square

Corollary 5.2. *Suppose that $Nil(R)$ is divided. Then either $Nil(R) = \{0\}$ (i.e., R is reduced) or 0 and 1 are the only idempotents of R .*

The next two results also hold for almost boldly radical ideals.

Proposition 5.3. *Suppose that R admits an almost strongly radical ideal J of R . Then $Nil(R)$ is a divided ideal.*

Proof. Let $a \in Nil(R)$ and $b \in R - Nil(R)$. Since $a^n \in b^nJ$ for some $n \geq 1$ and J is almost strongly radical, $a \in bJ$. Thus $Nil(R) \subset bJ \subset bR$. Hence $Nil(R)$ is divided. \square

In view of Propositions 5.1 and 5.3, we have the following three corollaries.

Corollary 5.4. *Suppose that $Nil(R)$ is an almost strongly radical ideal of R . Then either $Nil(R) = \{0\}$ (i.e., R is reduced) or $Nil(R)$ is a divided prime ideal.*

Corollary 5.5. *Let R be an almost strongly rooty ring. Then either $Nil(R) = \{0\}$ (i.e., R is reduced) or $Nil(R)$ is a divided prime ideal.*

Corollary 5.6. *Let R be a ring with $Nil(R) \neq \{0\}$. Then $Nil(R)$ is a divided prime ideal of R if and only if $Nil(R)$ is an almost strongly radical ideal of R .*

Proof. We need only prove (\Rightarrow) . Suppose that $a^n \in b^n Nil(R)$ for some $a \in R, b \in R - Nil(R)$, and $n \geq 1$. Then $a \in Nil(R)$ since $Nil(R)$ is a radical ideal. Since $Nil(R)$ is divided and $b \notin Nil(R)$, we have $a \in Nil(R) \subset bR$. Thus $a = br$ for some $r \in R$. Since $Nil(R)$ is prime by Proposition 5.1 and $b \notin Nil(R)$, we have $r \in Nil(R)$, and thus $a = br \in b Nil(R)$. Hence $Nil(R)$ is almost strongly radical. \square

Corollary 5.7. *The following statements are equivalent for a ring R .*

- (1) $Nil(R)$ is divided.
- (2) $Nil(R)$ is almost strongly radical.
- (3) R admits an almost strongly radical ideal.

Proof. (1) \Rightarrow (2) If $Nil(R) = \{0\}$, then $Nil(R)$ is certainly almost strongly radical. If $Nil(R) \neq \{0\}$, then $Nil(R)$ is almost strongly radical by Proposition 5.1 and Corollary 5.6. (2) \Rightarrow (3) is clear, and (3) \Rightarrow (1) follows by Proposition 5.3. \square

Proposition 5.8. *Suppose that $Nil(R)$ is a divided ideal of R . Then a radical ideal I of R is an almost strongly (resp., boldly) radical ideal of R if and only if $I/Nil(R)$ is a strongly (resp., boldly) radical ideal of $R/Nil(R)$.*

Proof. By Proposition 5.1, we may assume that $Nil(R)$ is a prime ideal. Let $N = Nil(R)$, $J = I/N$, and $D = R/N$. We give the proof for strongly radical ideals; the boldly radical case is left to the reader.

Suppose that I is an almost strongly radical ideal of R . Suppose that $a^n + N \in (b^n + N)J$ for some $a \in R, b \in R - N$, and $n \geq 1$. Thus $a^n = b^n i + w$ for some $i \in I$ and $w \in N$. Since $b^n \notin N, w \in N$, and N is a divided prime ideal, $w = b^n d$ for some $d \in N$. Hence $a^n = b^n(i + d)$. Since $i \in I$ and $d \in N \subset I$, we have $i + d \in I$. Since I is an almost strongly radical ideal, $a = bj$ for some $j \in I$. Hence $a + N = (b + N)(j + N)$, and thus $a + N \in (b + N)J$.

Conversely, let I be a radical ideal of R such that $J = I/N$ is a strongly radical ideal of $D = R/N$. We need to show that I is an almost strongly radical ideal of R . Suppose that $a^n \in b^n I$ for some $a \in R, b \in R - N$, and $n \geq 1$. Hence $a^n + N \in (b^n + N)J$. Thus $a + N \in (b + N)J$ since J is a strongly radical ideal of D . Hence $a = bi + w$ for some $i \in I$ and $w \in N$. Since $b \notin N, w \in N$, and N is a divided prime ideal of R , we have $w = bd$ for some $d \in N$. Hence $a = b(i + d)$. Since $i + d \in I, a \in bI$. Thus I is an almost strongly radical ideal of R . \square

We next relate the two different ways to generalize the "closedness" concepts from integral domains to rings with zerodivisors. The first result is an immediate consequence of Corollary 5.5 and Proposition 5.8.

Proposition 5.9. *A ring R is almost strongly rooty if and only if $\text{Nil}(R)$ is divided and $R/\text{Nil}(R)$ is strongly rooty.*

Proposition 5.10. *A ring R is almost strongly (2,3)-closed if and only if $\text{Nil}(R)$ is divided and $R/\text{Nil}(R)$ is strongly (2,3)-closed.*

Proof. Let $N = \text{Nil}(R)$. Suppose that R is almost strongly (2,3)-closed. Let $a \in N$ and $b \notin N$. Then $b^n | a^n$ for all sufficiently large $n \geq 1$, and hence $b | a$. Thus $N \subset bR$, and hence N is divided. Suppose that $(b + N)^2 | (a + N)^2$ and $(b + N)^3 | (a + N)^3$ in R/N . We may assume that $b \notin N$. Then $cb^2 = a^2 + x$ and $db^3 = a^3 + y$ for some $c, d \in R$ and $x, y \in N$. Since N is divided, we have $x, y \in b^3R$. Thus $b^2 | a^2$ and $b^3 | a^3$ in R with $b \notin N$, so $b | a$ since R is almost strongly (2,3)-closed. Hence $(b + N) | (a + N)$, and thus R/N is strongly (2,3)-closed.

Conversely, suppose that $N = \text{Nil}(R)$ is divided and R/N is strongly (2,3)-closed. Suppose that $b^2 | a^2$ and $b^3 | a^3$ in R with $b \notin N$. Then $(b + N)^2 | (a + N)^2$ and $(b + N)^3 | (a + N)^3$ in R/N , and hence $(b + N) | (a + N)$ since R/N is strongly (2,3)-closed. Thus $cb = a + n$ for some $n \in N$. Since N is divided, we have $n \in bR$, and thus $b | a$. Hence R is almost strongly (2,3)-closed. \square

Proposition 5.11. *A ring R is almost strongly root closed (resp., n -root closed for some $n \geq 2$) if and only if $\text{Nil}(R)$ is divided and $R/\text{Nil}(R)$ is strongly root closed (resp., n -root closed).*

Proof. The proof is similar to that of Proposition 5.10, and hence will be omitted. \square

Recall from [17] that an integral domain R with quotient field K is called a *pseudo-valuation domain* (PVD) in case each prime ideal P of R is strongly prime, in the sense that $xy \in P, x \in K, y \in K$ implies that either $x \in P$ or $y \in P$. In [6], the study of pseudo-valuation domains was generalized to the context of arbitrary rings (possibly with nonzero zerodivisors). Recall from [6] that a prime ideal P of R is said to be *strongly prime* (in R) if aP and bR are comparable for all $a, b \in R$. If each prime ideal of R is strongly prime, then R is called a *pseudo-valuation ring* (PVR). Examples of PVRs include PVDs and chained rings (*i.e.*, any two ideals are comparable under inclusion), and a PVR is quasilocal [6, Lemma 1]. The two concepts agree for integral domains.

Recently, [9] gave another generalization of PVDs to the context of arbitrary rings. Recall from [18, Proposition 1.1(6)] that a ring R is a Φ -PVR if and only if $\text{Nil}(R)$ is a divided prime ideal of R and for each $a, b \in R - \text{Nil}(R)$, either $b | a$ or $a | bc$ for each nonunit $c \in R$ (for the original definition of Φ -PVRs, see [9]). Observe that a PVR is a Φ -PVR by [9, Corollary 7(3)], but a Φ -PVR need not be a PVR by [18, Theorem 2.6]. Also, observe that a PVD is a strongly rooty ring, and hence it is an almost strongly rooty ring. We have the following result.

Proposition 5.12. *Let R be a Φ -PVR. Then R is an almost strongly rooty ring.*

Proof. Since $\text{Nil}(R)$ is a divided prime ideal of R by [18, Proposition 1.1(6)], in light of Proposition 5.9 we need only show that $D = R/\text{Nil}(R)$ is a strongly rooty ring. Since for each $a, b \in R$, either $a | b$ or $b | ac$ for each nonunit c of R by [18, Proposition 1.1(6)], for each $d, e \in D$ either $d | e$ or $e | dm$ for each nonunit m of D . Hence D is a PVD by [6, Theorem 5]. Thus $D = R/\text{Nil}(R)$ is a strongly rooty ring. \square

Corollary 5.13. *Let R be a PVR. Then R is an almost strongly rooty ring. In particular, a chained ring is an almost strongly rooty ring.*

Proof. Since a PVR is a Φ -PVR by [9, Corollary 7(3)] and a chained ring is a PVR, the claim is now clear by Proposition 5.12. \square

Note that a Φ -PVR need not be almost strongly root closed since a PVD need not be root closed (for example, $R = \mathbb{R} + XC[X]$ is a PVD which is not root closed).

We next show that the concepts defined in this section are all equivalent if R is zero-dimensional. Recall that a ring R is π -regular if for each $a \in R$, there is a $b \in R$ and an integer $n \geq 1$ such that $a^n b a^n = a^n$. In fact, a ring is π -regular if and only if it is zero-dimensional [14, Theorem 3.1]. Thus a reduced ring is π -regular if and only if it is von Neumann regular.

Proposition 5.14. *The following statements are equivalent for a zero-dimensional ring R (i.e., a π -regular ring).*

- (1) *Either R is reduced or quasilocal (with maximal ideal $Nil(R)$).*
- (2) *$Nil(R)$ is divided.*
- (3) *R is almost strongly rooty.*
- (4) *R is almost strongly root closed.*
- (5) *R is almost strongly n -root closed for some integer $n \geq 2$.*
- (6) *R is almost strongly (2,3)-closed.*

Proof. If R is reduced, then $Nil(R) = \{0\}$ is divided and the statements (3)–(6) are equivalent by Proposition 4.6. Suppose that R is not reduced, i.e., $Nil(R) \neq \{0\}$. Then any of statements (1)–(6) implies that $Nil(R)$ is divided. Conversely, if $Nil(R)$ is divided, and hence prime by Proposition 5.1, then R is quasilocal. Hence $R/Nil(R)$ is a field; so statements (3)–(6) are all satisfied by Propositions 5.9, 5.10, and 5.11. Thus statements (1)–(6) are all equivalent if R is zero-dimensional. \square

Easy examples show that the above conditions need not be equivalent if $dim(R) \geq 1$.

Note that a finite ring R is always n -root closed, root closed, (2,3)-closed, and rooty since $T(R) = R$. However, by Corollary 4.7, a finite ring R is strongly n -root closed (for some $n \geq 2$), strongly root closed, strongly (2,3)-root closed, strongly rooty, or seminormal if and only if it is reduced (i.e., a direct product of finitely many finite fields).

The following results follow directly from Proposition 5.14, since a finite ring is zero-dimensional.

Corollary 5.15. *A finite ring R is almost strongly rooty (resp., almost strongly n -root closed (for some $n \geq 2$), almost strongly root closed, almost strongly (2,3)-closed) if and only if either R is reduced (i.e., a direct product of finitely many finite fields) or R is quasilocal.*

Corollary 5.16. *Let $n \geq 2$. Then \mathbb{Z}_n is almost strongly rooty (resp., almost d -root closed (for some $d \geq 2$), almost strongly root closed, almost strongly (2,3)-closed) if and only if either n is square-free or $n = p^m$ for some prime p and $m \geq 1$.*

Remark 5.17.

- (a) $R = \mathbb{Z}_4$ is almost strongly root closed (resp., almost strongly (2,3)-closed, almost strongly rooty). However, R is not reduced, and thus R is not strongly root closed (resp., strongly (2,3)-closed, strongly rooty).

- (b) Unlike the case for integral domains, observe that if each maximal ideal of R is an almost strongly radical ideal, then R need not be an almost strongly rooty ring by Example 4.4. (Note that the ring in Example 4.4 is reduced.)

In view of Remark 5.17(b), we have the following result.

Proposition 5.18. *Let R be a ring with $\text{Nil}(R) \neq \{0\}$. Then R is an almost strongly rooty ring if and only if each maximal ideal of R is an almost strongly radical ideal.*

Proof. Suppose that R is an almost strongly rooty ring. Then it is clear that each maximal ideal of R is an almost strongly radical ideal. Conversely, assume that each maximal ideal of R is an almost strongly radical ideal and $\text{Nil}(R)$ is nonzero. First observe that $\text{Nil}(R)$ is a prime ideal of R by Propositions 5.3 and 5.1. Let I be a radical ideal of R , and choose a maximal ideal M of R such that $I \subset M$. Suppose that $a^n \in b^n I$ for some $a \in R, b \in R - \text{Nil}(R)$, and $n \geq 1$. Thus $a^n = b^n i$ for some $i \in I$. Hence $a = bm$ for some $m \in M$, and thus $b^n(i - m^n) = 0$. We need to show that $m \in I$. Since $b^n \notin \text{Nil}(R)$ and $\text{Nil}(R)$ is prime, $i - m^n \in \text{Nil}(R) \subset I$. Hence $m^n \in I$, and thus $m \in I$. \square

A proof similar to that of Proposition 5.18 shows that if each maximal ideal of R is almost boldly radical and $\text{Nil}(R) \neq \{0\}$, then each radical ideal of R is almost boldly radical. The following result also follows from the proof of Proposition 5.18.

Corollary 5.19. *Let R be a ring such that $\text{Nil}(R)$ is a divided prime ideal of R . Let I and J be radical ideals of R such that $J \subset I$ and I is an almost strongly (resp., boldly) radical ideal of R . Then J is an almost strongly (resp., boldly) radical ideal of R .*

Our final result is the analog of Proposition 4.3 for almost strongly rooty rings. Recall that, even for integral domains, a quasilocal (almost strongly) rooty ring need not be (almost) strongly root closed.

Proposition 5.20. *Suppose that R is an almost strongly rooty ring. If R is not quasilocal, then R is an almost strongly root closed ring.*

Proof. The proof is similar to that of Proposition 4.3; we leave the details to the reader. \square

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REFERENCES

- [1] D.F. Anderson, "Root Closure in Integral Domains", *J. Algebra*, **79** (1982), pp. 51–59.
- [2] R.G. Swan, "On Seminormality", *J. Algebra*, **67** (1980), pp. 210–229.
- [3] D.E. Rush, "Seminormality", *J. Algebra*, **67** (1980), pp. 377–384.
- [4] R. Gilmer and R.C. Heitmann, "On $\text{Pic}(R[X])$ for R Seminormal", *J. Pure Appl. Algebra*, **16** (1980), pp. 251–257.
- [5] D.F. Anderson and A. Badawi, "Conditions Equivalent to Seminormality in Certain Classes of Commutative Rings", *Lecture Notes in Pure and Applied Mathematics*, **220** (2001), pp. 49–59. New York/Basel: Marcel Dekker.
- [6] A. Badawi, D.F. Anderson, and D.E. Dobbs, "Pseudo-Valuation Rings", *Lecture Notes in Pure and Applied Mathematics*, **185**, (1997), pp. 57–67. New York/Basel: Marcel Dekker.
- [7] D.F. Anderson, A. Badawi, and D.E. Dobbs, "Pseudo-Valuation Rings, II", *Boll. Un. Mat. Ital.*, **B(8) 3** (2000), pp. 535–545.
- [8] A. Badawi, "On Domains Which Have Prime Ideals That are Linearly Ordered", *Comm. Algebra*, **23** (1995), pp. 4365–4373.

- [9] A. Badawi, "On Φ -Pseudo-Valuation Rings", *Lecture Notes in Pure and Applied Mathematics*, **205** (1999), pp. 101–110. New York/Basel: Marcel Dekker.
- [10] D.D. Anderson and D.F. Anderson, "Multiplicatively Closed Subsets of Fields", *Houston J. Math.*, **13** (1987), pp. 1–11.
- [11] J. Sato and T. Sugatani, "On the Radical Ideals of Seminormal Rings", *Comm. Algebra*, **18** (1990), pp. 441–451.
- [12] D.F. Anderson and J. Park, "Rooty and Root Closed Domains", *Lecture Notes in Pure and Applied Mathematics*, **205** (1999), pp. 87–99. New York/Basel: Marcel Dekker.
- [13] R. Gilmer, *Multiplicative Ideal Theory*. New York/Basel: Marcel Dekker, 1972.
- [14] J. Huckaba, *Commutative Rings with Zero Divisors*. New York/Basel: Marcel Dekker, 1988.
- [15] I. Kaplansky, *Commutative Rings*, rev. edn. Chicago: Univ. of Chicago Press, 1974.
- [16] D.F. Anderson, "Root Closure in Commutative Rings: a Survey", *Lecture Notes in Pure and Applied Mathematics*, **205** (1999), pp. 55–71. New York/Basel: Marcel Dekker.
- [17] J.R. Hedstrom and E.G. Houston, "Pseudo-Valuation Domains", *Pacific J. Math.*, **75** (1978), pp. 137–147.
- [18] A. Badawi, "On Φ -Pseudo-Valuation Rings, II", *Houston J. Math.*, **26** (2000), pp. 473–480.

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